SOME PROPERTIES OF GROUP DIVISIBLE DESIGNS

by

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1. Introduction

An experimental design is a partially balanced incomplete block (in short, PBIB) design if there are b blocks such that

(a) each block contains $k \leq v$ different treatments, where k = number of units per block and v = number of treatments,

(b) each treatment occurs in r blocks, where r = number of replicates of each treatment, and

(c) any two treatments which are i-th associates occur together in λ_i blocks, $i = 1, 2, \ldots, m$.

In a degenerate case when $m = 1$, a PBIB design reduces to a balanced incomplete block (BIB) design.

These designs, introduced by Bose and Nair (1939) and developed by Rao (1948), Connor and Clatworthy (1954), are arranged in blocks or groups that are smaller than a complete replication, in order to eliminate heterogeneity to a greater extent than is possible with randomized blocks and latin squares. In planning experiments in the physical sciences, agriculture and genetics, one is often confronted with natural limitations on the size of experimental blocks. In order to allow more freedom of choice in the number of replicates, designs which lack the complete symmetry of the balanced designs must be used. Therefore, use of PBIB designs is becoming more widespread.

But PBIB designs are less suitable than BIB designs. The statistical analysis is more complicated. When the variation among blocks is large, some pairs of treatments are more precisely compared than others, and several different standard errors may have to be computed for tests of significance. These difficulties increase as the design departs more and more from the symmetry of the BIB design.

It has been shown by R. C. Bose and T. Shimamoto (1952) that all PBIB designs

with 2 associate classes, can be divided into five distinct types:

(a) Group Divisible (GD),

(b) Triangular (T) including the subtypes Triangular Doubly Linked Blocks (TDLB) and Triangular Singly Linked Blocks (TSLB),

(c) Singly Linked (SL),

(d) Latin Squares (LS) with i constraints (L_i), and

(e) Cyclic (C).

One simple and important type is GD designs. An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible if the treatments can be divided into m groups, each with n treatments, so that the treatments belonging to the same groups occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks. If $\lambda_1 = \lambda_2 = \lambda$ (say) then every pair of treatments occurs together in λ blocks and the design reduces to a BIB design. That is why a PBIB (or GD) design is a special case of a BIB design.

The object of this report is to review the more important properties of GD designs with two associate classes and to give an illustrative example. A special feature is the division of GD design into three types:

(a) Singular GD (SGD) design characterized by $r = \lambda_1$,

(b) Semi-regular GD (SRGD) design characterized by $r > \lambda_1$, $rk = v\lambda_2$, and

(c) Regular GD (RGD) design characterized by $r > \lambda_1$, $rk > v\lambda_2$.

These three types are described in more detail in the subsequent sections. For practical purposes, the useful range will be confined to $r \leq 10$, $k \leq 10$, and λ_i $(i = 1, 2)$ chosen not to exceed 3, except for a few special cases.

2. Association Scheme And Relations Among PBIB Designs Parameters

The necessary notations concerning a PBIB design's association scheme and relations among their parameters are briefly given below. Reference should be made to Cochran and Cox (1950), Bose (1951, 1963), and Ogawa (1959).

(1) Association Scheme. Given v treatments, $t_1, t_2, \ldots, t_v$, a relation among them satisfying the following three conditions is called an association with m associate classes:

(a) Any two treatments are either first or second, ..., m-th associates,

(b) Each treatment has n_i i-th associates, $i = 1, 2, \ldots, m$, and

(c) For each pair of treatments which are i-th associates, there are p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$ treatments which are j-th associates of the one treatment of the pair and the same time k-th associates of the other.

(2) PBIB Designs With Relations Among Their Parameters. For a PBIB design based on any association scheme, the parameters

$$v, n_i, p^i_{jk} \ (i, j, k = 1, 2, \ldots, m), \tag{2.1}$$

may be called parameters of the first kind, and the additional parameters

$$b, r, k, \lambda_i \ (i = 1, 2, \ldots, m) \tag{2.2}$$

may be called parameters of second kind. Clearly

$$vr = bk, \tag{2.3}$$

$$n_1 + n_2 + \ldots + n_m = v - 1, \tag{2.4}$$

$$n_1\lambda_1 + n_2\lambda_2 + \ldots + n_m\lambda_m = r(k - 1). \tag{2.5}$$

By definition the number p^i_{jk} is independent of which pair t_1, t_2 of i-th associates as above-mentioned. Consider the pair t_1, t_2, then

$$p^i_{jk} = p^i_{kj}. \tag{2.6}$$

The following relations shown by Bose and Nair (1939) are:

$$\sum_{k=1}^{m} p^i_{jk} = n_j \qquad \text{if } i \neq j \tag{2.7}$$

$$\sum_{k=1}^{m} p_{jk}^{i} = n_j - 1, \qquad \text{if } i = j, \tag{2.8}$$

$$n_i p_{jk}^{i} = n_j p_{ik}^{j} = n_k p_{ij}^{k}. \tag{2.9}$$

(3) Relations Among The Parameters For GD Designs. In GD design with 2 associate classes, Bose and Connor (1952) have shown the following relations and inequalities between the parameters v, b, r, k, m, n, λ_1, λ_2,

$$v = nm, \qquad n_1 = n - 1, \qquad n_2 = n(m - 1), \tag{2.10}$$

$$vr = bk, \tag{2.11}$$

$$\lambda_1(n - 1) + \lambda_2 n(m - 1) = r(k - 1), \tag{2.12}$$

$$r - \lambda_1 \geq 0, \qquad rk - v\lambda_2 \geq 0. \tag{2.13}$$

Nair (1943) reported the following important inequality which can be used to study the basic properties of five distinct types in a PBIB design with two association classes.

$$\Delta = (r - \lambda_1)\ (r - \lambda_2) + (\lambda_1 - \lambda_2)\left[(r - \lambda_1)p_{12}^2 - (r - \lambda_2)p_{12}^1\right] \tag{2.14}$$

is non-negative.

If $p_{12}^1 = 0$, then in the case of GD designs, (2.14) reduces to

$$\begin{aligned}\Delta &= (r - \lambda_1)(r - \lambda_2) + (\lambda_1 - \lambda_2)(r - \lambda_1)p_{12}^2 \\ &= (r - \lambda_1)\left[(r - \lambda_2) + (\lambda_1 - \lambda_2)(n - 1)\right] \\ &= (r - \lambda_1)\left[r(k - 1) + r - nm\lambda_2\right] \\ &= (r - \lambda_1)(rk - v\lambda_2),\end{aligned} \tag{2.15}$$

if (2.10) and (2.12) are used.

If $p_{12}^2 = 0$, then (2.14) reduces to

$$\Delta = (r - \lambda_2)(rk - v\lambda_1), \tag{2.16}$$

Clearly $r \geq \lambda_1$ and $r \geq \lambda_2$. Further if $r > \lambda_1$ or $r > \lambda_2$ it follows that

$$rk \geq v\lambda_1 \quad \text{or} \quad v\lambda_2 \tag{2.17}$$

according as p_{12}^2 or $p_{12}^1 = 0$.

The parameters, p_{jk}^{i} $(i, j, k = 1, 2)$, of GD designs may be conveniently written in the form of two symmetric matrices

$$P_1 = (p_{jk}^{1}) = \begin{pmatrix} (n-2) & 0 \\ 0 & n(m-1) \end{pmatrix},$$

$$P_2 = (p_{jk}^{2}) = \begin{pmatrix} 0 & (n-1) \\ (n-1) & n(m-2) \end{pmatrix}. \tag{2.18}$$

For example, let $m = 4$, $n = 3$. The corresponding GD association scheme is

A B C D

E F G H

I J K L .

The first associates of the treatment A are E and I, and the second associates are B, C, D, F, G, H, J, K and L.

A PBIB design based on the above association scheme for which the parameters of the second kind are

$v = 12$, $b = 9$, $r = 3$, $k = 4$, $\lambda_1 = 0$ and $\lambda_2 = 1$, is shown below;

(A H B C), (H E F G), (E B K D),

(F K L A), (K J I H), (J L C E),

(L I G B), (G D A J), (I C D F).

3. The Incidence Matrix

It is helpful to use the incidence matrix to show some properties of GD designs in the following sections.

(1) General Meaning. An arrangement of a certain number of "treatments"

in a certain number of "block" in such a way that some prescribed combinatorial conditions are fullfilled is a statistical design. With every design is associated a unique matrix called the incidence matrix of the design.

Let N be the incidence matrix of a PBIB design with m associate classes and N' denote the transpose matrix of N. Then the determinant NN' may be written as

$$|NN'| = rk(r - z_1)^{\alpha_1} \dots (r - z_t)^{\alpha_t}, \tag{3.1}$$

$$\sum_{h=1}^{t} \alpha_h = v - 1, \qquad t \leq m, \tag{3.2}$$

where the z's are different, $r - z_h (h = 1, 2, \dots, t)$, are factors of NN' and $(h = 1, 2, \dots, t)$, are their respective multiplicities, Connor and Clatworthy (1954).

For m general it is observed for $v > b$ that $|NN'|$ is zero, which implies that one of the factors is zero, and for $v = b$ that $|NN'|$ is an integral square.

(2) Special Meaning For GD Designs. The value of $|NN'|$ for GD designs was first proposed by Bose and Connor (1952). It is necessary that

$$|NN'| = rk(rk - v\lambda_2)^{m-1}(r - \lambda_1)^{m(n-1)}. \tag{3.3}$$

To evalute (3.3), let N denote the incidence matrix with dimension vxb. Then

$$N = (n_{ij}) = \begin{pmatrix} n_{11} & n_{12} & \dots & n_{1b} \\ n_{21} & n_{22} & \dots & n_{2b} \\ \dots & \dots & \dots & \dots \\ n_{v1} & n_{v2} & \dots & n_{vb} \end{pmatrix}. \tag{3.4}$$

The rows correspond to treatments, the columns correspond to blocks and

$n_{ij} = 1$ or 0 according as the i-th treatment does or does not occur in the j-th block. It is easy to see that

$$\sum_{j=1}^{b} n_{ij} = \sum_{j=1}^{b} n_{ij}^2 = r, \tag{3.5}$$

$$\sum_{j=1}^{b} n_{ij}n_{uj} = \lambda_1 \text{ or } \lambda_2, \qquad i, u = 1, 2, \ldots, v, \tag{3.6}$$

according as the i-th and u-th treatments do or do not belong to the same group. Hence

$$NN' = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \cdot & \cdot & \cdots & \cdot \\ B & B & \cdots & A \end{bmatrix}, \tag{3.7}$$

where A and B are nxn matrices defined by

$$A = \begin{bmatrix} r & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & r & \cdots & \lambda_1 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1 & \lambda_1 & \cdots & r \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \end{bmatrix}. \tag{3.8}$$

Each row or column in the matrix (3.7) contains A in the diagonal position and contains B in the other $m - 1$ positions.

To evalute $|NN'|$ proceed as follows.

(a) Add the second, third, ..., nm-th rows in (3.7) to the first row, every element of first row becomes

$$r + (n - 1)\lambda_1 + n(m - 1)\lambda_2 = rk,$$

by (2.12).

(b) Take rk outside the determinant, and subtract the first row multiplied

by λ_2 from all other rows. Then

$$|NN'| = rk \begin{vmatrix} C & D & \cdots & D \\ 0 & E & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & E \end{vmatrix}, \qquad (3.9)$$

where C, D, E and O are the square matrix of n, and defined as:

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1-\lambda_2 & r-\lambda_2 & \cdots & \lambda_1-\lambda_2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \cdots & r-\lambda_2 \end{pmatrix}, \qquad D = \begin{pmatrix} I_{1xn} \\ 0_{(n-1)xn} \end{pmatrix},$$

$$E = \begin{pmatrix} r-\lambda_2 & \lambda_1-\lambda_2 & \cdots & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & r-\lambda_2 & \cdots & \lambda_1-\lambda_2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \cdots & r-\lambda_2 \end{pmatrix}. \qquad (3.10)$$

And 0_{nxn} is a null matrix. Therefore,

$$\begin{aligned} |NN'| &= rk|C||E|^{m-1} \\ &= rk(rk - v\lambda_2)^{m-1}|C||C|^{m-1} \\ &= rk(rk - v\lambda_2)^{m-1}(r - \lambda_1)^{m(n-1)}, \end{aligned}$$

using (2.10), (2.11) and (2.12), the result is obtained.

It is obvious that the quantity $rk - v\lambda_2$ occuring above is non-negative. It is necessary to prove this statement for the case $r > \lambda_1$.

Let N_1 be the submatrix formed from the matrix N given by (3.4), by taking the first 2n rows which correspond to the treatments of the first two groups.

Then

$$N_1N_1' = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \qquad (3.11)$$

where A and B are same as (3.8). Then

$$|N_1N_1'| = \{r + (n - 1)\lambda_1 + n\lambda_2\}(r - \lambda_1)^{2(n-1)}(rk - v\lambda_2) \,. \qquad (3.12)$$

Since $|N_1N_1'| \geq 0$, it follows that $rk - v\lambda_2 \geq 0$ if $r > \lambda_1$. Hence all GD designs can be divided into the three exhaustive and mutually exclusive types as mentioned in section 1.

4. Confounded Property For GD Designs

One of important properties of GD designs is that it is a special case of counfounded designs for asymmetrical factorial experiments involving two factors, Nair and Rao (1948). They discussed 2-factor experiments in detail showing the estimation of the treatment differences, efficiency (informations for variances and covariances), and tests of significance.

Table I

Treatment Combinations of a p x q Factorial

	Levels of Y				
Levels of X	x_1y_1	x_1y_2	...	...	x_1y_q
	x_2y_1	x_2y_2	...	...	x_2y_q
	.	.	...	...	.
	x_py_1	x_py_2	...	...	x_py_q

Let X and Y be any two factors with p levels and q levels respectively, and assume p is smaller than q. If the pq treatment combinations represented in Table I are laid out in b blocks of size k plots ($k < pq$) in such a way that

any two treatments in the same column occur together in λ_{10} blocks, any two treatments in the same row occur together in λ_{01} blocks and any two others occur together in λ_{11} blocks, then a PBIB design with three associate classes and the parameters are:

$$pqr = bk \tag{4.1}$$

$$n_1 = (p - 1), \qquad n_2 = (q - 1), \qquad n_3 = (p - 1)(q - 1) \tag{4.2}$$

$$(p - 1)\lambda_{10} + (q - 1)\lambda_{10} + (p - 1)(q - 1)\lambda_{11} = r(k - 1) \tag{4.3}$$

$$P_1 = (p^1_{jk}) = \begin{pmatrix} (p-2) & 0 & 0 \\ 0 & 0 & (q-1) \\ 0 & (q-1) & (p-2)(q-1) \end{pmatrix},$$

$$P_2 = (p^2_{jk}) = \begin{pmatrix} 0 & 0 & (p-1) \\ 0 & (q-2) & 0 \\ (p-1) & 0 & (p-1)(q-1) \end{pmatrix},$$

$$P_3 = (p^3_{jk}) = \begin{pmatrix} 0 & 1 & (p-2) \\ 1 & 0 & (q-2) \\ (p-2) & (q-2) & (p-2)(q-2) \end{pmatrix}. \tag{4.4}$$

The following inequalities proved by Nair and Rao (1948) are:

$$r \geq \lambda_{10} + (q - 1)(\lambda_{11} - \lambda_{01}) \tag{4.5}$$

$$r \geq \lambda_{01} + (p - 1)(\lambda_{11} - \lambda_{10}) \tag{4.6}$$

$$r \geq \lambda_{01} + \lambda_{10} - \lambda_{11}. \tag{4.7}$$

Now the following two special cases of confounded design are discussed:

Case A: Assuming $\lambda_{10} = \lambda_1$ and $\lambda_{01} = \lambda_{11} = \lambda_2$ the design becomes a PBIB design with two associate classes and the parameters,

$$v = pq, \qquad n_1 = (p - 1), \qquad n_2 = p(q - 1). \tag{4.8}$$

$$p^1_{jk} = \begin{pmatrix} (p - 2) & 0 \\ 0 & p(q - 1) \end{pmatrix},$$

$$p^2_{jk} = \begin{pmatrix} 0 & (p - 1) \\ (p - 1) & p(q - 2) \end{pmatrix}. \tag{4.9}$$

Since $p^1_{12} = 0$, this is a GD design with $m = q$ and $n = p$.

By the assumptions of case A, (4.5), (4.6) and (4.7) will be reduced, respectively, to:

$$r \geq \lambda_1 \tag{4.10}$$

$$rk \geq v\lambda_2 \tag{4.11}$$

$$r \geq \lambda_1 \tag{4.12}$$

Case B: Assuming $\lambda_{01} = \lambda_1$ and $\lambda_{10} = \lambda_{11} = \lambda_2$ the design once again becomes a PBIB design with two associate classes, the parameters of which can be obtained from those for case A, if p and q are interchanged. Since $p^1_{12} = 0$, this type is also a GD design with $m = p$ and $n = q$.

Similarly, by the assumptions of case B, (4.5), (4.6) and (4.7) can be reduced to:

$$rk \geq v\lambda_2 \tag{4.13}$$

$$r \geq \lambda_1 \tag{4.14}$$

$$r \geq \lambda_1 \tag{4.15}$$

From the inequalities (4.10) to (4.15), they actually are left with two fundamental inequalities, namely, $r \geq \lambda_1$ and $rk \geq v\lambda_2$ of which the first is obviuosly true for any incomplete design. The second inequality $rk \geq v\lambda_2$

is satisfied in all three types of GD designs. By definition, the relation $rk \geq v\lambda_2$ is true for SRGD and RGD designs. The relation $rk \geq v\lambda_2$ can be proved for SGD designs as follows:

The parameters of a BIB design will be denoted by a starred letter in order to distinguish them from the paramenters of GD designs. Since the relation $\lambda^*(v^* - 1) = r^*(k^* - 1)$ holds for a BIB design, then:

$$\begin{aligned} rk - v\lambda_2 &= n(r^*k^* - v^*\lambda^*) \\ &= n(r^* - \lambda^*) \\ &\geq 0. \end{aligned}$$

Since all the three inequalities (4.5), (4.6) and (4.7) can become equalities only if $r = \lambda_{10} = \lambda_{01} = \lambda_{11}$, i.e., if $k = pq$ or the design has complete blocks. For GD design in case A and B, $r = \lambda_1$ and $rk = v\lambda_2$ cannot be satisfied simultaneously. Therefore there exist only three admissible conditions forming three distinct types of GD design as shown in section 1.

Nair and Rao (1948) used the equation,

$$\begin{aligned} p_{11} &= (1/k)\{r(k - 1) - (\lambda_{11} - \lambda_{01} - \lambda_{10})\} \\ &= (1/k)\{p\lambda_{10} + q\lambda_{01} + (pq - p - q)\lambda_{11}\}, \end{aligned}$$

to show some confounding properties among main effects and interaction. In case A and B, if $r = \lambda_1$, then $p_{11} = r$, and the interaction XxY is unconfounded. Further, by (4.10) and (4.14) one of the main effects is not confounded (namely, of X in case A and of Y in case B). If $r > \lambda_1$, the interaction XxY and one of the main effects are confounded (X in case A and Y in case B). Similarly, when $rk = v\lambda_2$, then one of the main effects (Y in case A and X in case B) is not confounded. When $rk > v\lambda_2$, then one of the main effects (Y in case A and X in case B) is confounded.

The three types of GD designs have the following categories of confounding if it is considered as a quasi-factorial experiment involving two factors: one

at m levels and the other at n levels.

(a) One of the main effects and the interaction are not confounded.

(b) One of the main effects and the interaction are confounded, and

(c) Both the main effects and the interaction are confounded.

5. Singular GD Design

A GD design is said to be singular if $r = \lambda_1$. A SGD design is always derivable from a corresponding BIB design on replacing each treatment by a group of n treatments. These groups, in the SGD designs, give the groups of the association scheme. Thus the BIB design's plan, with parameters $v^* = b^* = 7$, $r^* = k^* = 3$, $\lambda^* = 1$, is given below, the column representing the blocks

A	B	C	D	E	F	G
F	G	A	B	C	D	E
E	F	G	A	B	C	D

Let $n = 3$, then the treatment A is replaced by A_1, A_2, A_3 and the same for the other treatments. It follows that the SGD design has the parameters $v = 21$, $b = 7$, $r = 3$, $k = 9$, $\lambda_1 = 3$, $\lambda_2 = 1$, $m = 7$, $n = 3$, the plan for which is shown below:

A_1	B_1	C_1	D_1	E_1	F_1	G_1
A_2	B_2	C_2	D_2	E_2	F_2	G_2
A_3	B_3	C_3	D_3	E_3	F_3	G_3
F_1	G_1	A_1	B_1	C_1	D_1	E_1
F_2	G_2	A_2	B_2	C_2	D_2	E_2
F_3	G_3	A_3	B_3	C_3	D_3	E_3
E_1	F_1	G_1	A_1	B_1	C_1	D_1
E_2	F_2	G_2	A_2	B_2	C_2	D_2
E_3	F_3	G_3	A_3	B_3	C_3	D_3

The association scheme is

$$\begin{array}{ccccccc} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & G_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 & G_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 & F_3 & G_3 \end{array}$$

In general, corresponding to the BIB design with the parameters v^*, b^*, r^*, k^*, λ^*, then there is a SGD design with the parameters,

$$v = nv^*, \quad b = b^*, \quad r = r^*, \quad k = nk^*, \quad \lambda_1 = r,$$
$$\lambda_2 = \lambda^*, \quad m = v^*, \quad n = n^*. \tag{5.1}$$

A useful class of SGD designs is obtained by starting with the unreduced balanced incomplete block design with the parameters,

$$v^* = t, \quad b^* = t(t - 1)/2, \quad r^* = t - 1, \quad k^* = 2, \quad \lambda^* = 1, \tag{5.2}$$

obtained by taking for blocks all possible pairs out of t treatments, and then replacing each treatment by n new treatments (n = 2, 3, 4, or 5). The resulting SGD design has the parameters,

$$v = nt, \quad b = t(t - 1)/2, \quad r = t - 1, \quad k = 2n,$$
$$\lambda_1 = t - 1, \quad \lambda_2 = 1, \quad m = t, \quad n = n. \tag{5.3}$$

Conversely consider a SGD design with the paraments, v, b, r, k, λ_1, λ_2, m and n where $r = \lambda_1$. Let t_1 and t_2 be any two treatments belonging to the same group. t_1 occurs in r blocks, and since $r = \lambda_1$, t_2 must occur in each of these r blocks and nowhere else. Hence if a treatment occurs in a certain blocks, every treatment belonging to the group occurs in that block. Let each group of treatments be replaced by a single treatment in the design, then there are $v^* = m$ treatments in the new design and because any two treatments belonging

to different groups occur together λ_2 times in the original design, the new design is a BIB design with parameters,

$$v^* = m, \quad b^* = b, \quad r^* = r = \lambda_1, \quad k^* = k/n, \quad \lambda^* = \lambda_2, \quad n^* = n. \tag{5.4}$$

The problem of constructing a SGD design, therefore, offers no difficulty. However, if r^* and λ^* differ too much, then in the derived GD design, the accuracy of the within group and between group comparisons will appreciable differ.

In a resolvable design, the blocks are separable into groups, each group (or replication) containing all variates, each variate occuring once and only once in the replication. Each replication must necessarily contain the same number of blocks, say t, so that,

$$v = tk, \quad b = tr. \tag{5.5}$$

Resolvable designs are of special important in analysis of variance. Here they are employed to understand the constructive properties of resolvable SGD design.

Table II

Parameters of some SGD Designs and the Corresponding BIB Designs from which They are Derivable

Designs	Parameters of BIB Design					Parameters of SGD Design							
Number	v^*	b^*	r^*	k^*	λ^*	v	b	r	k	m	n	λ_1	λ_2
1	4	6	3	2	1	12	6	3	6	6	3	3	1
2	4	6	3	2	1	16	6	3	8	6	4	3	1
3	5	10	4	2	1	10	10	4	4	5	2	4	1
4	5	10	4	2	1	15	10	4	6	5	3	4	1
5	7	7	3	3	1	14	7	3	6	7	2	3	1
6	7	7	3	3	1	21	7	3	9	7	3	3	1
7	9	12	4	3	1	18	12	4	6	9	2	4	1
8	13	13	4	4	1	26	13	4	8	13	2	4	1

A SGD design may be considered to belong to the designs as the corresponding BIB design. It is clear that if a BIB design is resolvable the same is true of a SGD design derived from it (shown in 1, 2, and 7 of Table II).

Table II gives some cases of practical interest. Theorems and methods of constructing GD designs can be referred to in Bose et al. (1953) and Bose et al. (1954).

Fisher (1940) has shown that a necessary condition for the existence of a BIB design with v* treatments and b* blocks is

$$b^* \geq v^*, \tag{5.6}$$

hence for a SGD design $b \geq m$. If possible, let $b < m$. Consider the mxm incidence matrix,

$$N_2 = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1b} & 0 & \cdots & 0 \\ n_{21} & n_{22} & \cdots & n_{2b} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ n_{b1} & n_{b2} & \cdots & n_{bb} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ n_{m1} & n_{m2} & \cdots & n_{mb} & 0 & \cdots & 0 \end{bmatrix}, \tag{5.7}$$

where the last $m - b$ columns of N_2 consist of zeros. Then,

$$|N_2N_2'| = \begin{vmatrix} r & \lambda_2 & \cdots & \lambda_2 \\ \lambda_2 & r & \cdots & \lambda_2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_2 & \lambda_2 & \cdots & r \end{vmatrix}$$

$$= \{r + \lambda_2(m - 1)\}(r - \lambda_2)^{m-1}$$

$$= (rk/n)(r - \lambda_2)^{m-1}, \tag{5.8}$$

from (2.12). But

$$|N_2N_2'| = |N_2||N_2'| = 0. \tag{5.9}$$

therefore, $r = \lambda_2$, and contradicts the fact $\lambda_1 \neq \lambda_2$ for GD designs. The assumption $b < m$ is incorrect and $b \geq m$ for SGD design.

6. Semi-regular GD Design

A GD design is said to be semi-regular if $r > \lambda_1$ and $rk - v\lambda_2 = 0$. Hence (2.12) reduces to

$$r + (n - 1)\lambda_1 = n\lambda_2. \tag{6.1}$$

Each block must contain the same number of treatments from each group so that k must be divided by m denoting

$$c = k/m \quad \text{or} \quad k = cm \tag{6.2}$$

as shown by Bose and Connor (1952).

For a SRGD design there holds the inequality

$$b \geq v - m + 1 \tag{6.3}$$

The value of (3.3) given by (3.7) is singular for the case of SRGD design. Its rank is not less than $v - m + 1$ by the following proof.

Let $R(NN')$ be the rank of matrix NN' defined as (3.7) and use the definition of the rank of a matrix (Fuller, 1962). From (3.9), strike out the last row and column from matrix E, then it becomes $(n-1)x(n-1)$ matrix F. By the definition of SRGD design and (6.1),

$$|C| = (r - \lambda_1)^{n-1}, \ |F| = (\lambda_2 - \lambda_1)(r - \lambda_1)^{n-2}.$$

If from (3.9), strike out the 2n-th, ..., mn-th rows and columns, then the

resulting matrix $|NN'|^*$ is

$$|NN'|^* = rk|C||F|^{m-1}$$
$$= rk(\lambda_2 - \lambda_1)^{m-1}(r - \lambda_1)^{mn-2n+1}$$
$$\neq 0,$$

since $\lambda_2 - \lambda_1 \neq 0$ and $r - \lambda_1 \neq 0$. But

$$R(NN') = R(NN')^* \leq b.$$

Hence

$$R(NN')^* \geq n+(m-1)(n-1)$$
$$= v - m + 1.$$

Then the resulting $b \geq v - m + 1$ for SRGD design is obtained.

For a resolvable SRGD design there holds the inequality

$$b \geq v - m + r. \tag{6.4}$$

From the definition of resolvable designs defined in (5.5), the blocks can be divided into r groups, of $b/r = t$ blocks each, such that each group of blocks gives a complete replication, then $R(NN') = R(NN')^* \leq b - r + 1$. Since in N the sum of the columns corresponding to a complete replication must give a column consisting of unities. Thus not more than $b - r + 1$ column vectors are independent. Hence the result $b \geq v - m + r$ is held for a resolvable SRGD design.

In forming the incidence matrix N for an affine resolvable SRGD design the blocks shall be arranged in such a way that the first t columns correspond to the blocks of the first replication; next t columns to those of the second replication and so on.

To the incidence matrix N adjoin m columns such that the j-th adjoined column contains 1's in the positions $(j-1)n+1$, $(j-1)n+2$, ..., $(j-1)n+n = jn$, and 0's elsewhere. Then, this new matrix is extended by joining r rows such that the i-th adjoined row will have 1's in the positions $(i-1)t+1$, $(i-1)t+2$, ..., $(i-1)t+t = it$, and 0's elsewhere. Let this augmented square matrix of order $(v+r)$ be denoted by N^* then

$$N^*N^{*\prime} = \begin{bmatrix} r+1 & \lambda_1+1 & \cdots & \lambda_1+1 & \lambda_2 & \cdots & \lambda_2 & 1 & 1 & \cdots & 1 \\ \lambda_1+1 & r+1 & \cdots & \lambda_1+1 & \lambda_2 & \cdots & \lambda_2 & 1 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \lambda_1+1 & \lambda_1+1 & \cdots & r+1 & \lambda_2 & \cdots & \lambda_2 & 1 & 1 & \cdots & 1 \\ \lambda_2 & \cdot & \cdots & \lambda_2 & r+1 & \cdots & \lambda_2 & 1 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \lambda_2 & \cdot & \cdots & \lambda_2 & \lambda_1+1 & \cdots & r+1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & \cdot & \cdot & \cdots & 1 & 1 & t & 0 & 0 \\ 1 & 1 & \cdots & \cdot & \cdot & \cdots & 1 & 1 & 0 & t & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & \cdots & \cdot & \cdot & \cdots & 1 & 1 & 0 & 0 & t \end{bmatrix}$$

Using (2.10), (2.11), (6.1), and $rk - v_2 = 0$, then

$$\begin{aligned} |N^*N^{*\prime}| = |N^*|^2 &= t^r n^m k^{m(n-1)} \\ &= v^r n^m k^{b-2r} \end{aligned} \qquad (6.5)$$

since $b = v - m + r$ and $v = tk$. (6.5) must be a perfect square. It can be stated as follows:

(a) With odd number of blocks in an affine resolvable SRGD design,

1. c must be a perfect square if m and r are odd.

2. nk must be a perfect square if m is odd and r is even.

3. t must be a perfect square if m is even and r is odd.

(b) With even number of blocks in an affine resolvable SRGD design,

1. n must be a perfect square if m is odd and r is even.

2. m must be a perfect square if m and r are odd.

Table III gives the parameters of some useful SRGD designs with $r \leq 6$.

Table III

Parameters of Some SRGD Designs with $r \leq 6$

Design number	Parameters v	b	r	k	m	n	λ_1	λ_2
1	6	8	4	3	3	2	0	2
2	9	9	6	6	3	3	3	4
3	10	8	4	5	5	2	0	2
4	12	9	3	4	4	3	0	1
5	12	12	6	6	3	4	2	3
6	14	8	4	7	7	2	0	2
7	18	12	6	9	9	2	0	3
8	20	16	4	5	5	4	0	1

Design 1 gives the following association scheme and plan where the columns represent the blocks.

Group

A	B	C
D	E	F

Plan

Rep. I		Rep. II		Rep. III		Rep. IV	
A	D	A	D	A	D	A	D
B	E	B	E	E	B	E	B
C	F	F	C	C	F	F	C

7. Regular GD Design

A GD design is said to be regular if $r > \lambda_1$ and $rk - v\lambda_2 > 0$, and to be regular symmetry if $b = v$ and in consequence $r = k$. Hence

$$
\begin{aligned}
|NN'| &= |N|^2 \\
&= r^2(r^2 - v\lambda_2)^{m-1}(r - \lambda_1)^{m(n-1)} \qquad (7.1)
\end{aligned}
$$

from (3.3). It follows that for a regular symmetical GD design:

(a) if m is even, then $r^2 - v\lambda_2$ is a perfect square,

(b) if m is odd and n is even, then $r - \lambda_1$ must be a perfect square.

For a RGD design there holds the inequality

$$b \geq v \qquad (7.2)$$

It is easy to prove using definition of rank. Then

$$v = R(NN') = R(N) \leq b$$

from (3.3), $|NN'| > 0$.

For a resolvable RGD there holds the inequality

$$b \geq v + r - 1. \qquad (7.3)$$

If the design is resolvable then as before (in section 6) $R(N) \leq b - r + 1$. Hence the result $b \geq v + r - 1$ exists.

Corresponding design 3 in Table IV it gives the following association scheme and plan where the columns represent the blocks

Group				Plan									
A	B	C	Rep. I	A	B	C	D	E	F	G	H	I
D	E	F	Rep. II	D	E	F	G	H	I	A	B	C
G	H	I	Rep. III	G	H	I	A	B	C	D	E	F
			Rep. IV	B	C	D	E	F	G	H	I	A .

Table IV gives the parameters of some useful RGD design with $r \leq 6$.

Table IV

Parameters of Some RGD Designs with $r \leq 6$

Designs Number	v	b	r	k	m	n	λ_1	λ_2
1	6	6	4	4	2	3	3	2
2	8	8	3	3	4	2	0	1
3	9	9	4	4	3	3	3	1
4	12	12	4	4	6	2	2	1
5	12	20	5	3	6	2	0	1
6	14	14	4	4	7	2	0	1
7	15	30	6	3	5	3	0	1
8	20	16	4	5	5	4	0	1

8. Illustrative Example

A numberical example of an analysis of variance of a GD design follows:

(1) Analysis With Intra-block Information. The parameters of design 4 given in Table III are

$$v = 12,\ b = 9,\ r = 3,\ k = 4,\ m = 4,\ n = 3,\ \lambda_1 = 0,\ \lambda_2 = 1. \tag{8.1}$$

For any PBIB design with two associate classes, Bose and Shimamoto (1952) defined four computational constants c_1, c_2, H and Δ by means of the following relations:

$$\begin{aligned}
k^2\Delta &= (rk-r+\lambda_1)(rk-r+\lambda_2)+(\lambda_1-\lambda_2)\left\{r(k-1)(p^1_{12}-p^2_{12})+\lambda_2 p^1_{12}-\lambda_1 p^2_{12}\right\} \\
kH &= (2rk-2r+\lambda_1+\lambda_2)+(p^1_{12}-p^2_{12})(\lambda_1-\lambda_2) \\
k\Delta c_1 &= \lambda_1(rk-r+\lambda_2)+(\lambda_1-\lambda_2)(\lambda_2 p^1_{12}-\lambda_1 p^2_{12}) \\
k\Delta c_2 &= \lambda_2(rk-r+\lambda_1)+(\lambda_1-\lambda_2)(\lambda_2 p^1_{12}-\lambda_1 p^2_{12}).
\end{aligned} \tag{8.2}$$

By (8.2), further parameters are

$$c_1 = 0, \; c_2 = 1/3, \quad H = 21/4, \quad \Delta = 27/4 \tag{8.3}$$

which are needed for the analysis of the design. The corresponding GD association scheme is

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12\,. \end{array} \tag{8.4}$$

Treatments in the same column of this scheme belong to the same group and are first associates. Treatments in different columns belong to different group and are second associates. It can be verified from the plan that treatments in the same column of this scheme do not occur together in a block, whereas treatments in different column occur together in just on block in accordance with the condition $\lambda_1 = 0$, $\lambda_2 = 1$. The various steps in the computations in the analysis of a GD design are the same for the three types. The analysis for design 4 (Table III) with parameters (8.1), (8.3), and association scheme (8.4) is given below.

Table V

Plan and Yields of Cotton (Pounds per Plot)

Block j	Treatments t				Block Totals B_j
1	(1) 2.6	(2) 2.1	(3) 2.3	(4) 2.8	9.8
2	(7) 2.8	(10) 2.5	(5) 2.7	(4) 3.2	11.2
3	(6) 2.7	(11) 2.3	(9) 2.4	(4) 3.2	10.7
4	(1) 2.7	(7) 2.9	(6) 4.1	(8) 2.5	12.2
5	(11) 2.5	(5) 2.7	(2) 2.5	(8) 3.1	10.8
6	(10) 2.9	(9) 2.7	(3) 2.4	(8) 3.2	11.2
7	(1) 2.8	(11) 2.6	(10) 2.6	(12) 2.7	10.7
8	(9) 3.2	(2) 2.2	(7) 3.0	(12) 3.4	11.8
9	(5) 2.8	(3) 2.8	(6) 2.6	(12) 3.3	11.5
					G = 99.9

Therefore, this experimental has 12 treatments arranged in blocks of size 4. The data, obtained from North Carolina Agricultural Experiment Station (1954), are the pounds of seed cotton per plot in a uniformity trial. The plan and yields are shown in Table V.

It is useful to divide the computations into a number of steps:

(a) Find the block totals B and the grand total G and insert them in the plan (Table VI).

(b) For each treatment, find the treatment total T and the total B_t of all blocks in which the treatment occurs. For treatment 1

$$T = 2.6 + 2.7 + 2.8 = 8.1$$

$$B_t = 9.8 + 12.2 + 10.7 = 32.7.$$

As a check, the sum of the T's is G and sum of the B_t's is kG. The T's and B_t's are written in the first two columns of the working Table VI.

Table VI

Computations for the Intra-block Analysis of a GD Design

Treat-ments	T	B_t	Q	G'	$108\hat{t}$	Adjusted Means
1	8.1	32.7	-0.3	- 1.5	- 2.1	2.756
2	6.8	32.4	-5.2	- 3.1	-59.3	2.226
3	7.5	32.5	-2.5	- 5.5	-24.5	2.548
4	9.3	31.7	5.5	10.1	55.9	3.293
5	8.2	33.5	-0.7	- 1.5	- 6.9	2.711
6	9.4	34.4	3.2	- 3.1	41.5	3.159
7	8.7	35.2	-0.4	- 5.5	0.7	2.781
8	8.8	34.2	1.0	10.1	1.9	2.793
9	8.3	33.7	-0.5	- 1.5	- 4.5	2.733
10	8.0	33.1	-1.1	- 3.1	-10.1	2.681
11	7.4	32.2	-2.6	- 5.5	-25.7	2.537
12	9.4	34.0	3.6	10.1	33.1	3.081
Total	99.9	399.6	0.0	0.0	0.0	

(c) From these two columns, form a third column of the values

$$Q = kT - B_t.$$

The values should sum exactly to zero.

(d) The treatment means $\hat{t}$, adjusted for block effects, can now be computed. The formula is

$$rk(k - 1)\hat{t} = (k - c_2)Q + (c_1 - c_2)S_1(Q). \tag{8.5}$$

The quantity $S_1(Q)$ denotes the sum of Q, taken over the first associates of the treatment in question. Thus in finding the adjusted mean of treatment 1, add the Q values for treatments 5 and 9, which are the first associate of treatment 1. For this design, let

G' = Sum of Q over all treatments in the same row in the association scheme.

For treatment 1, G' is the sum of the Q over treatment 1, 5 and 9. Then it follows that

$$G' = S_1(Q) + Q = -0.3 - 0.7 - 0.5 = -1.5.$$

Substituting for $S_1(Q)$ in terms of G' in the equation (8.5) for $\hat{t}$, then

$$\begin{aligned} rk(k - 1)\hat{t} &= (k - c_2 - c_1 + c_2)Q + (c_1 - c_2)G' \\ &= (k - c_1)Q + (c_1 - c_2)G'. \end{aligned} \tag{8.6}$$

Finally, substituting numberical values for the constants in (8.6) for $\hat{t}$,

$$108\hat{t} = 12Q - G'.$$

The values of $108\hat{t}$ appear in column (5) of Table VI. They should add to zero.

(e) To obtain the adjusted treatment means, multiply the values in column

(5) by 0.009259, the reciprocal of 108, and add the general mean, 2.775. For treatment 1 this gives

$$(0.009259)(-2.1) + 2.775 = 2.756.$$

(f) In the analysis of variance, the total sum of squares and the sum of squares for blocks are found in usual way. The general formula for the adjusted treatments sum of squares is

$$(1/k)\sum \hat{t}Q = (1/4)\sum \hat{t}Q$$

where the sum is taken over all treatments. This may be written

$$(1/4)\sum \hat{t}Q = (1/432)(108\hat{t})Q = \frac{\text{Sum of products of columns (5) and (3)}}{432}.$$

Table VII

Intra-block Analysis of Variance

Source of Variations	Degree of Freedom		Sum of Squares	Mean of Squares
Block(unadj.)	b-1	8	0.9950	
Treatment(adj.)	v-1	11	2.3525	$0.2139 = E_t$
Error	vr-v-b+1	16	2.0600	$0.1288 = E_e$
Total	vr-1	35	5.4075	

The observed F-ratio of 1.66 (0.2139/0.1288) with 11 and 16 degrees of freedom is not significant at the 5% level.

For two treatments that are first associates, the error variance of the difference between their adjusted means is

$$\frac{2E_e}{r} \times \frac{(k - c_1)}{(k - 1)} = \frac{2(0.1288)}{3} \times \frac{(4 - 0)}{(4 - 1)} = 0.1145 .$$

For second associates, the error variance of the difference between the adjusted means is

$$\frac{2E_e}{r} \times \frac{(k - c_2)}{(k - 1)} = \frac{2(0.1288)}{3} = \frac{(4 - 1/2)}{(4 - 1)} = 0.1049 \ .$$

Since each treatment has $n_1 = n - 1$, first associates and $n_2 = n(m - 1)$, second associates, the average variance is

$$\frac{(n - 1)(0.1145) + n(m - 1)(0.1049)}{v - 1} = 0.1.66 \ .$$

The least signicant differences (LSD) for testing the difference between two treatments is obtained by multiplying the square root of the estimated variance with the values of t at the significance level desired and with vr-b-v+1 degrees of freedom. Thus for first associates,

$$\text{LSD } 5\% = 2.120\sqrt{0.1145} = 0.7174 \ ,$$

for second associates,

$$\text{LSD } 5\% = 2.120\sqrt{0.1049} = 0.6867 \ .$$

If it is desired to use the same approximate LSD 5% for every pair of treatments irrespective of whether they are first or second associates, then an average variance 0.1066 is

$$\text{Average LSD } 5\% = 2.120\sqrt{0.1066} = 0.6922 \ .$$

(2) Analysis With Recovery Of Inter-block Information. The computations made in part (1) are needed for this analysis. The methods are:

(a) Find the unadjusted treatments sum of squares in the usual way. This

comes to 2.5542. From this value and the analysis in Table VII, obtain the adjusted blocks sum of squares by the relation:

$$\text{Block (adj.)} = \text{block(unadj.)} + \text{treatment (adj.)} - \text{treatment (adj.)}$$
$$= 0.9950 + 2.3525 - 2.5542$$
$$= 0.7933.$$

Thus the following Table VIII is obtained.

Table VIII

Auxiliary Table for Inter-block Analysis of Variance

Souce of Variations	Degrees of Freedom		Sum of Squares	Mean of Squares
Block(adj.)	b-1	8	0.7933	$0.0992 = E_b$
Treatment (unadj.)	v-1	11	2.5542	
Error	vr-v-b+1	16	2.0600	$0.1288 = E_e$
Total	vr-1	35	5.4075	

(b) Various weighting coefficients are now computed, since they are designed to give component series an importance in proper relation to their real significance.

$$w = 1/E_e = 7.76$$

$$w' = \frac{v(r-1)}{k(b-1)E_b - (v-k)E_e} = 11.19$$

$$W = \frac{w'}{w - w'} \quad ;$$

It is necessary that coefficients d_1 and d_2 which are computational constants in analysis with inter-block information and take the place of the c_1 and c_2 in the intra- block analysis. These d's depend on W, on the c's and on other structural constants Δ, H, λ_1, and λ_2. by (8.3), then

$$d_1 = \frac{c_1 + r_1W}{\Delta + rHW + r^2W^2} = 0$$

$$d_2 = \frac{c_2 + r_2W}{\Delta + rHW + r^2W^2} = -0.1475.$$

(c) To replace the Q, then compute values P

$$P = w'B_t + wQ - (w'kG/v)$$
$$= 11.19B_t + 7.76Q - 372.627.$$

The p's are shown in column (1) of Table IX. For treatment 1

$$P = (11.19)(32.7) + (7.76)(-0.3) - 372.627$$
$$= -9.042.$$

The P's add to zero, apart from rounding errors.

(d) The adjusted treatment means $\hat{t}$'s are given by the equation

$$kr\{w' + w(k - 1)\}\hat{t}' = (k - d_2)P + (d_1 - d_2)S_1(P)$$

where S_1 denotes summation over the first associates of the treatment in question. For this design the formula can be written, as (8.5) of the intra-block analysis,

$$kr\{w' + w(k - 1)\}\hat{t}' = (k - d_2)P + (d_1 - d_2)G''$$

where the G" for any treatments is the sum of the P's over the row in which the treatment appears in the association scheme at the beginning of this section. The G" value are placed in column (2) of Table IX. For treatment 1

$$G'' = -9.042 - 3.194 + 0.596 = -11.640.$$

Hence

$$\hat{t}' = \frac{(k - d_2)}{kr\{w' + w(k - 1)\}} P + \frac{(d_1 - d_2)}{kr\{w' + w(k - 1)\}} G''$$

$$= 0.0100P + 0.0003G'' .$$

Table IX

Combined Intra- and Inter-block Estimates of Treatment Effects

Treat-ment	(1) P	(2) G"	(3) $\hat{t}'$	(4) Mean(adj.)
1	- 9.042	-11.640	-0.0939	2.681
2	-50.423	-24.056	-0.5114	2.264
3	-28.352	-42.680	-0.2963	2.479
4	24.776	78.376	0.2713	3.046
5	- 3.194	-11.640	-0.0354	2.740
6	37.141	-24.056	0.3642	3.140
7	18.157	-42.680	0.1688	2.944
8	17.831	78.376	0.2018	2.977
9	0.596	-11.640	0.0025	2.778
10	-10.774	-24.056	-0.1150	2.660
11	-32.485	-42.680	-0.3377	2.437
12	35.769	78.376	0.3812	3.156
	0.000	0.000		

Then, the adjusted treatment means are

$$\frac{G}{vr} + \hat{t}'$$

where G/vr is the general mean. For this example,

$$\text{Adjusted mean} = 2.775 + 0.0100P + 0.0003G'' .$$

These values appear in column (4) of Table IX.

(e) For the combined intra- and inter-block analysis there is no exact test of the hypothesis of the quantity of treatment means.

The estimated variance of the difference between two treatments which are first associates is

$$\frac{2(k - d_1)}{r\{w' + w(k - 1)\}} = 0.0774 \ .$$

Likewise the estimated variance of the difference between two treatments which are second associates is

$$\frac{2(k - d_2)}{r\{w' + w(k - 1)\}} = 0.0802 \ .$$

As in the intra-block analysis, an average variance of the difference between two treatment (adjusted) means is computed as

$$\frac{2(0.0774) + 9(0.0802)}{11} = 0.0797 \ .$$

Thus for first associates and second associates respectively,

$$\text{LSD } 5\% = 2.120\sqrt{0.0774} = 0.5899$$

$$\text{LSD } 5\% = 2.120\sqrt{0.0802} = 0.6008 \ .$$

Thus an average LSD is

$$\text{LSD } 5\% = 2.120\sqrt{0.0797} = 0.5985 \ .$$

(3) Summarization Of Results

The results of the analysis may be summarized in two tables. Column (2) of Table X gives the unadjusted treatment means obtained by dividing each treatment total T(taken from column (1) of Table VI) by r. Column (3) gives the adjusted means without recovery of inter-block information, obtained from column (6) of Table VI. Column (4) gives the adjusted means with recovery of inter-block information, obtained from column (4) of Table IX.

Table XI gives the variance of the estimated difference between two treatment means, and also the LSD values. The value of F-ratio for testing the hypothesis of the equality of treatment means is 1.66 with 11 and 16 degrees of freedom.

Table X

Treatment Means

Treatment	Unadjusted mean	Adjusted, with inter-block inf. Without recovery	Recovery
1	2.700	2.756	2.681
2	2.267	2.226	2.264
3	2.500	2.548	2.479
4	3.100	3.293	3.046
5	2.733	2.711	2.740
6	3.133	3.159	3.140
7	2.900	2.781	2.944
8	2.933	2.793	2.977
9	2.767	2.733	2.778
10	2.667	2.681	2.660
11	2.467	2.537	2.437
12	3.133	3.018	3.156

Table XI

Variance and LSD for Estimated Treatment Difference

Item	Intra-block	Combined Intra- and Inter-block
Variance of estimated treatment difference		
1. between 1st associates	0.1145	0.7174
2. between 2nd associates	0.1049	0.6867
3. average	0.1066	0.6922
LSD 5 per cent		
1. for 1st associates	0.0774	0.5899
2. for 2nd associates	0.0802	0.6008
3. for average	0.0797	0.5985 .

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SOME PROPERTIES OF GROUP DIVISIBLE DESIGNS

by

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An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible if the treatments can be divided into m groups each with n treatments so that the treatments of the same group occur together in λ_1 blocks and the treatments of different group occur together in λ_2 blocks, where $\lambda_1 \neq \lambda_2$. This report discusses some properties of group divisible designs with two associate classes.

For all group divisible designs the following relationships hold: $v = mn$, $bk = vr$, $\lambda_1(n - 1) + \lambda_2 n(m - 1) = r(k - 1)$, $r \geq \lambda_1$, and $rk \geq v\lambda_2$. It can be shown that these designs fall into three classes: (1) Singular GD, for which $r = \lambda_1$, (2) Semi-regular GD, for which $r > \lambda_1$, $rk = v\lambda_2$, and (3) Regular GD, for which $r > \lambda_1$, $rk > v\lambda_2$. Each type of group divisible design has different confounded properties for asymmetrical factorial experiments.

A Singular GD design is always derivable from a balanced incomplete block design by replacing each treatment by a group of n treatments. When $b = v$ the quantity $(r - \lambda_1)^{m(n-1)}(rk - v\lambda_2)^{m-1}$ must be a perfect square shown by the incidence matrix N ($|N|^2 = |NN'|$). For Singular GD design $b \geq m$, for Regular GD design $b \geq v$, and for Semi-regular GD design $b \geq v - m + 1$, every block contains the same number of treatments from each group. It is also shown that for resolvable Regular GD design $b \geq v + r - 1$ and for resolvable Semi-regular GD design $b \geq v - m + r$.

The analysis of variance for intra- and inter-block estimated treatment means and the least significant difference at the 5 per cent level for testing estimated treatment difference has been given for sample data.